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ON THE SIMILARITY PROBLEM FOR POLYNOMIALLY BOUNDED OPERATORS ON HILBERT SPACE

BY

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ABSTRACT

Partial solutions are obtained to Halmos' problem, whether or not any polynomially bounded operator on a Hilbert space H is similar to a contraction. Central use is made of Paulsen's necessary and sufficient condition, which permits one to obtain bounds on $\|S\|\|S^{-1}\|$, where S is the similarity. A natural example of a polynomially bounded operator appears in the theory of Hankel matrices, defining

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix}$$

on $l^2 \oplus l^2$, where S is the shift and Γ_f the Hankel operator determined by f with $f' \in \mathrm{BMOA}$. Using Paulsen's condition, we prove that R_f is similar to a contraction. In the general case, combining Grothendieck's theorem and techniques from complex function theory, we are able to get in the finite dimensional case the estimate

$$\|S\|\|S^{-1}\| \leqq M^4 \log(\dim H)$$

where STS^{-1} is a contraction and assuming $\|p(T)\| \leqq M\|p\|_\infty$ whenever p is an analytic polynomial on the disc.

1. Introduction

A well-known theorem due to Von Neumann [17] asserts that if T is a contraction on a complex Hilbert space H, then the inequality

$$(1) \qquad \|p(T)\| \leqq \|p\|_\infty \equiv \sup_{z \in D} |p(z)|$$

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holds wherever $p(z) = \Sigma a_j z^j$ is an analytical polynomial on the disc $D = \{z \in \mathbf{C}; |z| \leqq 1\}$. If T is similar to a contraction, which means that STS^{-1} is a contraction for some invertible operator S on H, clearly

$$\|p(T)\| \leqq \|S\| \|S^{-1}\| \|p\|_\infty \leqq M \|p\|_\infty. \tag{2}$$

A problem of P. Halmos [5] is the question whether the converse property holds, i.e. if (2) is valid (T is then said to be polynomially bounded), T is similar to a contraction.

It was proved by Sz.-Nagy [15] that if T satisfies $\sup_{k \in \mathbf{Z}} \|T^k\| < \infty$, T is similar to a contraction. Rota [14] proved that every operator with spectral radius less than 1 is similar to a contraction.

We prove the following two results in this paper.

THEOREM 1. *Let R_f be the operator on $l^2 \oplus l^2$ given by*

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix}$$

where S is the shift and Γ_f the Hankel operator corresponding to f with $f(0) = 0$, $f' \in$ BMOA.

Then there exists an invertible operator A on $H = l^2 \oplus l^2$ such that AR_fA^{-1} is a contraction and $\|A\| \|A^{-1}\| \leqq C \|f'\|_{\mathrm{BMOA}}$.

This operator R_f was introduced by R. Rochberg in [13] and provides natural counter examples for several questions in operator theory, for instance, power bounded operators which are not polynomially bounded (cf. [11]). In [12] it was shown that R_f is polynomially bounded when $f' \in$ BMOA (this condition is only known to be sufficient). Theorem 1 is an improvement of this result (cf. [12], Q5).

THEOREM 2. *Let T be a polynomially bounded operator on an n-dimensional Hilbert space H, i.e., satisfying condition* (2) *with some bound M. Then there exists an invertible operator S on H such that STS^{-1} is a contraction and $\|S\| \|S^{-1}\| \leqq M^4 \log n$.*

This is the main result in the paper. Its proof is based on Grothendieck's inequality for bilinear forms and methods of Carleson measures.

The main part of this work was done while the author was visiting Odense University. The papers of V. Paulsen on the similarity problem were brought to his attention by U. Haagerup during this time.

2. Paulsen's similarity theorem

In [9], Paulsen proves that if A is a unital operator algebra (here A is the disc algebra $A(D)$) and ρ a completely bounded, unital homomorphism of A into the algebra of bounded operators on Hilbert space $B(H)$, then there exists a similarity S with $\|S\|\|S^{-1}\|=\|\rho\|_{cb}$ and such that $S^{-1}\rho(.)S$ is a completely contractive homomorphism.

It follows that $T\in B(H)$ is similar to a contraction iff there is a constant K so that

$$(3)\qquad \left|\sum_{1\leqq i,j\leqq n}\langle\varphi_{ij}(T)x_i,y_j\rangle\right|\leqq K\|(\varphi_{ij})_{1\leqq i,j\leqq n}\|_{M_n(A)}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{1/2}\left(\sum_{j=1}^{n}\|y_j\|^2\right)^{1/2}$$

whenever $(\varphi_{ij})_{1\leqq i,j\leqq n}$ is an $(n\times n)$ matrix of analytic polynomials ($n=1,2,\dots$) and $\{x_i\}_{i=1}^{n}$, $\{y_j\}_{j=1}^{n}$ are vectors in H. Here we define

$$\|(\varphi_{ij})\|_{M_n(A)}=\sup_{\substack{\Sigma|s_i|^2\leqq 1\\ \Sigma|t_j|^2\leqq 1}}\sup_{|z|<1}\left|\sum s_it_j\varphi_{ij}(z)\right|.$$

More precisely, there is an invertible $S\in B(H)$ such that $\|S\|\|S^{-1}\|\leqq K$ and $S^{-1}TS$ is a contraction.

Both Theorems 1 and 2 will be derived from this result.

3. The operator R_f

For $0<p<\infty$, let $H^p(D)\equiv H^p$ denote the Hardy space of analytic functions f on D satisfying

$$\|f\|_{H^p}\equiv\sup_{0<r<1}\left(\int|f(re^{i\theta})|^pd\theta\right)^{1/p}<\infty.$$

These functions have radial limit a.e. and H^p may therefore be viewed as a subspace of $L^p(\mathbf{T})$, $\mathbf{T}=$ circle.

Under the usual duality $\langle f,g\rangle=\int_{\pi}f\bar{g}$, the dual of H^1 identifies with the space BMOA of analytic functions on D with boundary values in the space BMO of functions of bounded mean oscillation. (More details on these matters can be found in [4].) Factoring H^1-functions as the product of H^2-functions, BMOA is identified with a subspace of the compact operators on H^2, the Hankel matrices. We use the standard notation

$$\Gamma_f=\sum\hat{f}(i+j)e_i\otimes e_i\quad\text{for } f\in\text{BMOA},\qquad \hat{f}(k)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta})e^{-ik\theta}d\theta.$$

In [13], R. Rochberg considered the operator

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ 0 & S \end{pmatrix}$$

acting on $l^2 \oplus l^2$, where S is the shift operators. Using the fact that

$$S^*\Gamma_f = \Gamma_f S = \Gamma_{S^* f}$$

the action of an analytic polynomial φ on R_f is easily computed:

$$\varphi(R_f) = \begin{pmatrix} \varphi(S^*) & \Gamma_{\varphi'(S^*)f} \\ 0 & \varphi(S) \end{pmatrix}.$$

R_f is power bounded iff f' is in the Bloch-class ([11]). Next we repeat the argument appearing in [12] to show that $f' \in \mathrm{BMOA} \Rightarrow R_f$ is polynomially bounded. Assume $f(0) = 0$. We need to prove

$$|\langle f, \varphi' h\rangle \leqq C \| f' \|_{\mathrm{BMOA}} \| \varphi \|_\infty \| h \|_{H^1} \tag{4}$$

for φ an analytic polynomial and $h \in H^1$.

Define the "square-function"

$$S(\alpha)(\theta) = \left(\int_{\Gamma_\theta} | \alpha'(z) |^2 dx dy \right)^{1/2}$$

where Γ_θ is the Stolz-angle in the point $\theta \in \mathbf{T}$.

Recall that if f is analytic on D, then $\| S(f) \|_1 \asymp \| f \|_{H^1}$. Since $f(0) = 0$, it follows from the H^1-BMO duality that

$$| \langle f, h\varphi' \rangle | \leqq C \| f' \|_{\mathrm{BMOA}} \cdot \int \left(\int_{\Gamma_\theta} | h\varphi' |^2 dx dy \right)^{1/2} d\theta.$$

Writing $h\varphi' = (h\varphi)' - h'\varphi$, the second factor estimates as $\| h\varphi \|_{H^1} + \| \varphi \|_\infty \| h \|_{H^1}$, implying (4).

4. Proof of Theorem 1

For $T = R_f$, Paulsen's condition on complete boundedness reduces to the inequality

$$\left| \left\langle f, \sum_{i,j} \varphi'_{i,j} x_i y_j \right\rangle \right| \leqq K \| (\varphi_{i,j}) \|_{M(A)} \left(\sum \| x_i \|_2^2 \right)^{1/2} \left(\sum \| y_i \|_2^2 \right)^{1/2} \tag{5}$$

where $\varphi_{i,j} \in A(D)$ and $x_i, y_j \in H^2$. Here $M(A)$ stands for $M_n(A)$; see Section 2. Again we may estimate (denoting $dxdy$ by dz)

$$\left|\left\langle f, \sum \varphi'_{i,j} x_i y_j \right\rangle\right| \leqq C \|f'\|_{\mathrm{BMOA}} \int \left(\int_{\Gamma_\theta} \left| \sum \varphi'_{i,j} x_i y_j \right|^2 dz \right)^{1/2} d\theta$$

and write

$$\sum \varphi'_{i,j} x_i y_j = \left(\sum \varphi_{i,j} x_i y_j \right)' - \sum \varphi_{i,j} x'_i y_i - \sum \varphi_{i,j} x_i y'_j. \tag{6}$$

Clearly $\Sigma\, \varphi_{i,j} x_i y_j$ is in H^1 and

$$\left\| \sum \varphi_{i,j} x_i y_j \right\|_{H^1} \leqq \| \varphi_{i,j} \|_{M(A)} \int_{\mathbf{T}} \left(\sum |x_i|^2 \right)^{1/2} \left(\sum |y_j|^2 \right)^{1/2}.$$

Hence, by the Cauchy–Schwartz inequality, we may evaluate the contribution of the first term in (6) by

$$\int \left(\int_{\Gamma_\theta} \left| \left(\sum \varphi_{i,j} x_i y_j \right)'(z) \right|^2 dz \right)^{1/2} d\theta \leqq C \| (\varphi_{i,j}) \|_{M(A)} \left(\sum \|x_i\|_2^2 \right)^{1/2} \left(\sum \|y_j\|_2^2 \right)^{1/2}.$$

Denoting $\|(\varphi_{i,j})\|_{M(A)} \equiv M$ for convenience, estimate

$$\begin{aligned} \int \left(\int_{\Gamma_\theta} \left| \sum \varphi_{i,j} x'_i y_j \right|^2 dz \right)^{1/2} d\theta &\leqq M \int \left(\int_{\Gamma_\theta} \left(\sum |x'_i|^2 \right) \left(\sum |y_j|^2 \right) dz \right)^{1/2} d\theta \\ &\asymp M \iint \left(\int_{\Gamma_\theta} |X'_\varepsilon(z)|^2 |Y_\delta(z)|^2 dz \right)^{1/2} d\theta\, d\varepsilon\, d\delta \end{aligned} \tag{$*$}$$

denoting $X_\varepsilon(z) = \Sigma\, \varepsilon_i x_i(z)$ and $Y_\delta(z) = \Sigma\, \delta_j y_j(z)$.

(The (ε_i) and (δ_j) refer to independent sequences of Rademacher functions on the Cantor group $\{1, -1\}^{\mathbf{N}}$.)

At this point, we invoke the following lemma, the proof of which is momentarily postponed.

Lemma 1. *There is an absolute constant C satisfying*

$$\int \left(\int_{\Gamma_\theta} |f'(z)|^2 |g(z)|^2 dz \right)^{1/2} d\theta \leqq C \|f\|_{H^2} \|g\|_{H^2} \tag{7}$$

for $f, g \in H^2(D)$.

This lemma permits one to majorize $(*)$ by $(f = X_\varepsilon,\ g = Y_\delta)$

$$CM \int \|X_\varepsilon\|_{H^2} \|Y_\delta\|_{H^2} d\varepsilon\, d\delta \leqq CM \left(\sum \|x_i\|_2^2 \right)^{1/2} \left(\sum \|y_j\|_2^2 \right)^{1/2}.$$

Together with the previous estimate, this gives (5) with $K = \|f'\|_{\mathrm{BMOA}}$.

PROOF OF LEMMA 1. The left member is dominated by $\int S(f)(\theta)g^*(\theta)d\theta$, where $g^*(\theta)\equiv\sup_{z\in\Gamma_\theta}|g(z)|$ denotes the nontangential function. By Cauchy–Schwartz and well-known estimates on harmonic functions (see[4]), we may further majorize by

$$\|S(f)\|_2 . \|g^*\|_2 \leqq C\|f\|_{H^2}\|g\|_{H^2}.$$

5. Use of Grothendieck's theorem

N. Varopoulos and S. Peller obtained estimations on $\|p(T)\|$ when T is a power bounded operator on a Hilbert space H, say

$$\sup\|T^n\|\leqq M$$

and p an analytical polynomial of a certain degree $d(p)$. The following inequality

$$\|p(T)\|\leqq M^2\log d(p).\sup_{z\in D}|p(z)| \tag{8}$$

can be derived from Grothendieck's fundamental theorem (see [7], p. 68, for instance). An example due to Davie (see [11] for details) shows that (8) is the best one may hope for.

The purpose of this section is to apply the same method to get a bound on the number K appearing in (3), involving $\max_{i,j} d(\varphi_{i,j})$. The next lemma is the first step in the proof of Theorem 1.

LEMMA 2. *Suppose $T\in B(H)$ satisfies $\sup_n\|T^n\|\leqq M$. Then*

$$\begin{aligned}&\left|\sum\langle\varphi_{i,j}(T)x_i, y_j\rangle\right|\\ &\leqq C\left(\max_{i,j}\log d(\varphi_{i,j})\right).M^2.\|(\varphi_{i,j})\|_{M(A)}.\left(\sum\|x_i\|^2\right)^{1/2}\left(\sum\|y_j\|^2\right)^{1/2}\end{aligned} \tag{9}$$

where $d(\varphi)$ is the degree of the polynomial φ.

PROOF. Denote $\{W_\lambda\}$ a system of diadic de-la-Vallée–Poussin type kernels ($\|W_\lambda\|_1\leqq 3$) supported by consecutive diadic intervals in $\mathbf{Z}_+$ and forming a partition of unity,

$$\sum_\lambda|\hat{W}_\lambda(n)|=1=\sum_\lambda\hat{W}_\lambda(n)\qquad\text{for } n\in\mathbf{Z}_+.$$

Write $\varphi^\lambda_{i,j}=\varphi_{i,j}*W_\lambda$. Clearly (9) reduces to proving for fixed λ

$$\left|\sum\langle\varphi^\lambda_{i,j}(T)x_i, y_j\rangle\right|\leqq CM^2\|(\varphi_{i,j})\|_{M(A)}\left(\sum\|x_i\|^2\right)^{1/2}\left(\sum\|y_j\|^2\right)^{1/2} \tag{10}$$

Since $W_\lambda \in H^1$, we may factorize $W_\lambda = g.h$ where $g, h \in H^2$ and $\|g\|_2\|h\|_2 \leqq 3$;

$$g(\theta) = \sum c_k e^{ik\theta}, \qquad h(\theta) = \sum d_l e^{il\theta}.$$

Then

$$\hat{\varphi}^{\lambda}_{i,j}(n) = \sum_{k+l=n} c_k d_l \hat{\varphi}_{i,j}(k+l)$$

and

$$\sum \langle \varphi^{\lambda}_{i,j}(T)x_i, y_j\rangle = \sum_{i,j,k,l} \|x_i\|\|y_j\| c_k d_l \hat{\varphi}_{i,j}(k+l) \langle T^k x'_i, (T^*)^l y'_j\rangle \tag{11}$$

denoting

$$x'_i = x_i\|x_i\|^{-1} \quad \text{and} \quad y'_j = y_j\|y_j\|^{-1}.$$

Since the $T^k x'_i$, $(T^*)^l y'_j$ in H satisfy

$$\|T^k x'_i\| \leqq M, \qquad \|(T^*)^l y'_j\| \leqq M$$

it follows from Grothendieck's inequality that

$$(11) \leqq K_G M^2 \sup \left| \sum_{i,j,k,l} \|x_i\|\|y_j\| c_k d_l \hat{\varphi}_{i,j}(k+l) s_{i,k} t_{j,l} \right|$$

where the supremum is taken over all 1-bounded scalar systems $\{s_{i,k}\}$ and $\{t_{j,l}\}$. Now

$$\sum \|x_i\|\|y_j\| c_k d_l \hat{\varphi}_{i,j}(k+l) s_{i,k} t_{j,l}$$

$$= \int_{\mathbf{T}} \left\{ \sum_{i,j} \|x_i\| \left(\sum_k c_k s_{i,k} e^{-ik\psi} \right) \|y_j\| \left(\sum_l d_l t_{j,l} e^{-il\psi} \right) \varphi_{i,j}(e^{i\psi}) \right\} m(d\psi)$$

can be estimated by

$$\|(\varphi_{i,j})\|_{M(A)} \cdot \int \left\{ \sum_i \|x_i\|^2 \left| \sum_k c_k s_{i,k} e^{-ik\psi} \right|^2 \right\}^{1/2} \left\{ \sum_j \|y_j\|^2 \left| \sum_l d_l t_{j,l} e^{-il\psi} \right|^2 \right\}^{1/2} m(d\psi)$$

which is bounded by

$$\|(\varphi_{i,j})\|_{M(A)} \cdot \left(\sum |c_k|^2 \right)^{1/2} \left(\sum |d_l|^2 \right)^{1/2} \left(\sum \|x_i\|^2 \right)^{1/2} (\|y_j\|^2)^{1/2}$$

from where the lemma follows.

6. Interpolation in the disc by polynomials of low degree

If T is an operator on an n-dimensional Hilbert space, T asssumed power bounded, its spectrum Spec(T) consists of at most n points in the closed unit disc

$\bar{D}$. Our purpose is to prove (3) replacing the $\varphi_{i,j}$ by analytic functions coinciding with $\varphi_{i,j}$ on Spec T and constructed piecing together polynomials of degree N ($\log N \sim \log n$) composed with a suitable conformal transformation of D. This will allow one to get (3) from inequalities as proved in Lemma 2. In this section, we present the required result on interpolating a finite set of points in D by a polynomial of small degree. The problem of piecing together will be presented in the next two sections.

The following lemma gives a rough estimation on polynomials obtained by Lagrange interpolation.

LEMMA 3. *Let $(a_j)_{1 \leq j \leq n}$ be distinct points in the disc $\{z \in \mathbf{C}; |z| < 1 - \varepsilon\}$ ($\varepsilon < \frac{1}{2}$) and denote for $f \in A(D)$ by Lf the unique polynomial of degree $n-1$ satisfying*

$$Lf(a_j) = f(a_j) \qquad \text{for } j = 1, \ldots, n.$$

Clearly L is a linear operator. Moreover

$$\| Lf \|_\infty = \varepsilon^{-2n^2} \sup_{|z| < 1-\delta} (f(z)) \qquad \text{where } \delta = \varepsilon / n^2.$$

PROOF. Clearly $L = L_n$ where inductively

$$L_1 f = f(a_1),$$

$$L_k f = L_{k-1} f + \prod_{j<k} \frac{z - a_j}{a_k - a_j} [f(a_k) - L_{k-1} f(a_k)].$$

Since $f - L_{k-1} f$ vanishes on $\{a_1, \ldots, a_{k-1}\}$, we may factorize

$$(f - L_{k-1} f)(z) = \prod_{j<k} \frac{z - a_j}{1 - \bar{a}_j z} \cdot \varphi(z). \tag{12}$$

Hence

$$L_k f = L_{k-1} f + \prod_{j<k} \frac{z - a_j}{1 - \bar{a}_j \cdot a_k} \varphi(a_k)$$

and

$$\| L_k f \|_\infty \leqq \| L_{k-1} f \|_\infty + 2^{k-1} \prod_{j<k} (1 - |a_j|)^{-1} |\varphi(a_k)|.$$

By (12) and a simple computation (since $\delta \ll \varepsilon$)

$$|\varphi(a_k)| \leqq \sup_{|z|=1-\delta} |\varphi(z)| \leqq \Big(\sup_{|z| \leq 1-\delta} |f(z)| + \| L_{k-1} f \|_\infty \Big) e^{C(k/\varepsilon)\delta}.$$

Therefore

$$\|L_k f\|_\infty \leqq \|L_{k-1} f\|_\infty + 2^{k-1}\varepsilon^{-k}|\varphi(a_k)| \leqq \varepsilon^{-2k} e^{C(k/\varepsilon)\delta}\Big(\|L_{k-1} f\|_\infty + \sup_{|z|\leqq 1-\delta}|f(z)|\Big)$$

and iterating

$$\|Lf\|_\infty \leqq \varepsilon^{-n^2} e^{C(\delta/\varepsilon)n^2} \sup_{|z|\leqq 1-\delta}|f(z)|$$

from where the lemma follows.

LEMMA 4. *Assume $(a_j)_{1\leqq j\leqq n}$ are points in the disc $\{z\in\mathbf{C}; |z|<1-\varepsilon\}$. Taking $N=\varepsilon^{-2}n^4$, there is a linear operator $I: A(D)\to\mathcal{P}_N$ into the space of polynomials of degree N, satisfying*

(i) $\|I\|_{\infty\to\infty}\leqq 3$,

(ii) *$If\,(a_j)=f(a_j)$ $(1\leqq j\leqq n)$.*

PROOF. Consider a kernel K verifying $\|K\|_{L^1(\pi)}\leqq 2$, $\hat{K}(j)\geqq 0$ and

$$\hat{K}(j)=1 \quad \text{if } |j|<N/2; \qquad \hat{K}(j)=0 \quad \text{if } |j|>N.$$

Let L be the operator of Lagrange interpolation for the set $\{a_1,\dots,a_n\}$ discussed above and take

$$If=(f*K)+L[f-(f*K)]$$

thus satisfying (i). Also, $\delta=\varepsilon n^{-2}$,

$$\begin{aligned}\|If\|_\infty &\leqq 2\|f\|_\infty+\varepsilon^{-n^2}\sup_{|z|<1-\delta}\sum_{j\geqq N/2}|\hat{f}(j)||z|^j\\ &\leqq\Big[2+\varepsilon^{-n^2}\sum_{j\geqq N/2}(1-\delta)^j\Big]\|f\|_\infty\\ &\leqq 3\|f\|_\infty\end{aligned}$$

by the choice of N.

7. Carleson measures and Beurling type function systems

As usual, if μ is a positive measure on the disc D, denote

$$\|\mu\|_C=\sup\frac{\mu(R(I))}{|I|}$$

where the supremum is taken over all intervals I in the circle $\mathbf{T}$ and $R(I)$ defined as

$$R(I)=\{z\in D; z/|z|\in I \text{ and } 1-|z|<\tfrac{1}{10}|I|\}.$$

LEMMA 5. *If $\{f_\alpha\}$ is a finite set of H^1 functions, then*

$$\int_D \max_\alpha |f_\alpha(z)|\mu(dz) \leqq C\|\mu\|_C \int_T \max_\alpha |f_\alpha(e^{i\theta})|\,d\theta.$$

PROOF. For $\alpha \in L^1(\pi)$, denote $\tilde{\alpha}$ the harmonic extension of α and α^* the non-tangential maximal function; thus

$$\tilde{\alpha}(z) = \alpha * P_z \ (P_z = \text{Poisson kernel at } z) \quad \text{and} \quad \alpha^*(\theta) = \sup_{z \in \Gamma_\theta} \tilde{\alpha}(z).$$

It follows from the factorization of an H^1-function as the product of a Blaschke-product and the square of an H^2-function that

$$f \in H^1 \Rightarrow |f(z)|^{\frac{1}{2}} \leqq \widetilde{|f|^{1/2}}(z) \qquad \text{for } z \in D.$$

Hence, defining $F(\theta) = \max_\alpha |f_\alpha(e^{i\theta})|^{1/2}$, we have

$$\max_\alpha |f_\alpha(z)| \leqq \tilde{F}(z)^2.$$

By Carleson's theorem (cf. [4]), it follows now that

$$\int \max_\alpha |f_\alpha(z)|\mu(dz) \leqq \int |\tilde{F}(z)|^2 \mu(dz) \leqq C\|\mu\|_C \int_{\mathbf{T}} |F^*(\theta)|^2 d\theta \leqq C\|\mu\|_C \|F\|^2_{L^2(\mathbf{T})}$$

proving the lemma.

REMARK. Lemma 5 may be reformulated as follows. Let $(X, \|\ \|)$ be a normed space and f a function in the vector valued Hardy space $H^1_X(D)$. Then

$$\int_D \|f(z)\|\mu(dz) \leqq C\|\mu\|_C \|f\|_{H^1_X}.$$

Indeed, it suffices to take $\{\langle f, x^*\rangle; x^* \in X^*, \|x^*\| \leqq 1\}$ for the system (f_α) (cf. also [3]).

LEMMA 6. *Let S be a finite set in the disc D. Suppose $S_\alpha \subset S$ and $S_\alpha \subset D_\alpha$ where the D_α are simply connected disjoint open subsets of D with disjoint rectifiable boundaries Γ_α. Denote $\Gamma = \bigcup_\alpha \Gamma_\alpha$ and μ the arc-length measure of Γ. Let $S^\alpha \subset S \backslash \bar{D}_\alpha$. There are H^∞ functions (φ_α) satisfying the following conditions:*

(i) *$\varphi_\alpha(z) = 1$ if $z \in S_\alpha$ and $\varphi_\alpha(z) = 0$ if $z \in S^\alpha$,*

(ii) *$\Sigma|\varphi_\alpha(z)| \leqq C\beta^{-1}\|\mu\|_C$ where*

$$\beta = \inf_{\alpha; z \in \Gamma_\alpha} \prod_{a \in S_\alpha \cup S^\alpha} \left| \frac{z-a}{1-\bar{a}z} \right|.$$

PROOF. Denote

$$B_\alpha(z) = \prod_{a \in S_\alpha} \frac{a - z}{1 - \bar{a}z} \quad \text{and} \quad B^\alpha(z) = \prod_{a \in S^\alpha} \frac{a - z}{1 - \bar{a}z}.$$

Suppose $\psi_\alpha \in H^\infty$ satisfy

$$\psi_\alpha(z) = B^\alpha(z)^{-1} \qquad \text{for } z \in S_\alpha. \tag{13}$$

The general solution of (13) is then given by

$$\eta_\alpha = \psi_\alpha + B_\alpha h_\alpha; \qquad h_\alpha \in H^\infty.$$

Hence, by duality

$$\inf \left\| \sum_\alpha |\eta_\alpha| \right\|_\infty = \sup \sum_\alpha \left| \int_{\mathrm{T}} \frac{\psi_\alpha(s)}{B_\alpha(s)} f_\alpha(s) ds \right| \tag{14}$$

where the inf is taken over all solutions of (13) and the sup ranges over the H^1 systems verifying the condition

$$\left\| \max_\alpha |f_\alpha| \right\|_{L^1(\mathrm{T})} \leqq 1. \tag{15}$$

Since B_α^{-1} has no poles outside D_α and $(B^\alpha)^{-1}$ no poles inside $\bar{D}_\alpha$, we may write by Cauchy's theorem

$$\int_{\mathrm{T}} \frac{\psi_\alpha(s)}{B_\alpha(s)} f_\alpha(s) ds = \int_{\Gamma_\alpha} \frac{f_\alpha(s)}{B_\alpha(s) B^\alpha(s)} ds$$

and the sum in (14) is therefore dominated by

$$\sum_\alpha \beta^{-1} \int_{\Gamma_\alpha} |f_\alpha(s)| ds \leqq \beta^{-1} \int_D \max_\alpha |f_\alpha(z)| \mu(dz).$$

We now invoke Lemma 5 and (15) to bound (14) by $C\beta^{-1}\|\mu\|_C$. Once the functions $\{\eta_\alpha\}$ are obtained with the property

$$\sum |\eta_\alpha(z)| \leqq C\beta^{-1}\|\mu\|_c \qquad \text{for } z \in D$$

the φ_α are given by multiplication with B^α, thus $\varphi_\alpha = \eta_\alpha B^\alpha$. Condition (ii) remains preserved and (i) follows from (13).

8. Cutting the disc

Denote

$$d(a,b) \equiv \left| \frac{a - b}{1 - \bar{a}b} \right|$$

the pseudo-distance for $a, b \in D$. We use the result of previous sections to prove the following.

LEMMA 7. *Let $S \subset D$ be a set of n distinct points. There are functions $\{\tau_\alpha\}$ in $A(D)$ and conformal maps Λ_α of D satisfying*

(i) $\Sigma\, \tau_\alpha(z) = 1$ *if* $z \in S$,

(ii) $\Sigma |\tau_\alpha(z)| \leqq \text{const.}$ *for* $z \in D$,

(iii) $|\Lambda_\alpha(z)| < 1 - n^{-6}$ *if* $\tau_\alpha(z) \neq 0$.

PROOF. Partition D in disjoint regions of the form

$$R_{k,\theta} = \{z \in D; n^{-k} \leqq 1 - |z| < n^{-k+1}, |\operatorname{Arg} z - \theta| < n^2 n^{-k}\}$$

and consider the subfamily $\{R_\alpha\}$ of those regions for which $S_\alpha = S \cap R_\alpha \neq \emptyset$. Write $\alpha \leftrightarrow k$ if R_α is an $R_{k,\theta}$. Define

$$D_\alpha = \{z \in D; d(z, R_\alpha) < 1 - 1/n\}; \qquad S^\alpha = \{z \in S; d(z, D_\alpha) > 1 - 1/n\}.$$

Notice that by definition of the $R_{k,\theta}$, the index set of α can be partitioned in 6 subsets for which the $\bar{D}_\alpha$ are mutually disjoint.

For $z \in \Gamma_\alpha \equiv \partial D_\alpha$

$$\prod_{a \in S_\alpha \cup S^\alpha} \left| \frac{z - a}{1 - \bar{a} z} \right| \geqq \prod_{a \in S, d(a,z) \geqq 1 - 1/n} d(a, z) \geqq (1 - 1/n)^n > c.$$

Fix an interval $I \subset \mathbf{T}$ and let $n^{-k'} < |I| \leqq n^{-k'+1}$. Denote λ the arc-length measure. Then

$$\lambda(\cup \Gamma_\alpha \cap R(I)) \leqq \sum_{k > k'-2} \sum_{\alpha \leftrightarrow k} \lambda(\Gamma_\alpha \cap R(I))$$

$$\leqq C|I| + C \sum_{k > k'+2} n^{2-k} . n \leqq C . |I|.$$

For fixed α, it follows from the construction that there is a point $a_\alpha \in D$ such that

$$d(z, a_\alpha) < 1 - n^{-6} \qquad \text{wherever } z \in S \backslash S^\alpha.$$

Thus the conformal map

$$\Lambda_\alpha(z) = \frac{z - a_\alpha}{1 - \bar{a}_\alpha z}$$

maps $S \backslash S^\alpha$ inside the disc $[|z| < 1 - n^{-6}]$. Application of Lemma 6 gives functions $\varphi_\alpha \in H^\infty$ with the properties

$$\varphi_\alpha(z)=1 \quad \text{if } z\in S_\alpha \quad \text{and} \quad \varphi_\alpha(z)=0 \quad \text{if } z\in S^\alpha,$$

$$\sum|\varphi_\alpha|\leqq C_1.$$

Since $S = \cup S_\alpha$, $\Pi(1-\varphi_\alpha)=0$ on S and we may write by expanding the product

$$1=\sum \phi_\alpha(z)\varphi_\alpha(z) \qquad \text{for } z\in S \quad \text{where } \phi_\alpha\in H^\infty, \quad |\phi_\alpha|<e^{C_1}.$$

Let $\tau_\alpha=\phi_\alpha\varphi_\alpha$. Clearly (i), (ii) hold and

$$z\in S, \qquad \tau_\alpha(z)\neq 0 \Rightarrow |\Lambda_\alpha(z)|<1-n^{-6}.$$

9. Proof of Theorem 2

Replacing T by $T_\varepsilon\equiv(1-\varepsilon)T$, an operator with spectrum contained in D is obtained, still satisfying $\|p(T_\varepsilon)\|\leqq M\|p\|_\infty$. Since, by a compactness argument, it suffices to obtain the similarity (with fixed bound) for T_ε, we may as well suppose $S\equiv\operatorname{Spec} T\subset D$. As a consequence of Lemma 7 above, there are functions $\{\tau_\alpha\}$ in $A(D)$ and conformal mappings Λ_α s.t.

$$\sum \tau_\alpha(z)=1 \qquad \text{on } s,^{\dagger} \tag{16}$$

$$\sum|\tau_\alpha|<C_1 \qquad \text{on } D, \tag{17}$$

$$|\Lambda_\alpha(z)|<1-n^{-6} \qquad \text{if } \tau_\alpha(z)\neq 0. \tag{18}$$

Fix α. From Lemma 4 and (18) a linear operator $I_\alpha: A\to\mathcal{P}_N$ $(N=n^{4+12})$ is obtained such that for $\psi\in A$

$$(I_\alpha\psi\circ\Lambda_\alpha)(z)=(\psi\circ\Lambda_\alpha)(z) \quad \text{for } z\in S, \qquad \tau_\alpha(z)\neq 0.$$

For $\varphi\in A(D)$,

$$\varphi=\sum \tau_\alpha(I_\alpha(\varphi\circ\Lambda_\alpha^{-1})\circ\Lambda_\alpha) \qquad \text{on } S$$

and we may therefore write

$$\varphi(T)=\sum \tau_\alpha(T)p_\alpha(\varphi)(T_\alpha) \tag{19}$$

denoting $p_\alpha(\varphi)=I_\alpha(\varphi\circ\Lambda_\alpha^{-1})$, $T_\alpha=\Lambda_\alpha(T)$.

Write by factorization $\tau_\alpha=\tilde{\tau}_\alpha\tilde{\tilde{\tau}}_\alpha$ with $|\tilde{\tau}_\alpha|^2=|\tau_\alpha|=|\tilde{\tilde{\tau}}_\alpha|^2$ on $\mathbf{T}$. Notice also

$$\sup_k\|\Lambda_\alpha(T)^k\|\leqq M.$$

Given a matrix (φ_{ij}) in A, it now follows from Lemma 2 and (19) that

† In case of multiplicity $j>1$ of some $z\in S$, the conditions $\Sigma\tau_\alpha^{(i)}(z)=0$ for $1\leqq i\leqq j$ must be added.

$$\left|\sum_{i,j}\langle\varphi_{i,j}(T)x_i, y_j\rangle \leqq \sum_\alpha \left|\sum_{i,j}\langle[p_\alpha(\varphi_{i,j})](T_\alpha)\tilde{\tau}_\alpha(T)x_i, \tilde{\tilde{\tau}}_\alpha(T)^* y_j\rangle\right|\right.$$

$$\text{(20)} \qquad \leqq C(\log n)M^2\sum_\alpha \|p_\alpha(\varphi_{i,j})\|_{M(A)}\Big(\sum_i \|\tilde{\tau}_\alpha(T)x_i\|^2\Big)^{1/2}\Big(\sum_j \|\tilde{\tilde{\tau}}_\alpha(T)^* y_j\|^2\Big)^{1/2}.$$

Since $\|p_\alpha(\varphi_{i,j})\|_{M(A)} \leqq \|I_\alpha\| \|(\varphi_{i,j}\circ\Lambda_\alpha^{-2}\|_{M(A)} \leqq C\|(\varphi_{i,j})\|_{M(A)}$

$$(20) \leqq M^2(\log n)\|(\varphi_{i,j})\|_{M(A)}\Big(\sum_i\sum_\alpha \|\tilde{\tau}_\alpha(T)x_i\|^2\Big)^{1/2}\Big(\sum_i\sum_\alpha \|\tilde{\tilde{\tau}}_\alpha(T)^*(y_j)\|^2\Big)^{1/2}.$$

For any $x \in H$, the map $u_x : A \to l^2$, $u_x(p) = p(T)x$ is in operator norm bounded by $\|x\|.M$. Hence, by [2], Corollary 2.8, $\pi_2(u_x) \leqq C_2\|x\|$ where C_2 is an absolute constant. This means that

$$\text{(21)} \qquad \Big(\sum \|\varphi_\alpha(T)x\|^2\Big)^{1/2} \leqq CM\|x\| \left\|\Big(\sum|\varphi_\alpha|^2\Big)^{1/2}\right\|_\infty$$

for $(\varphi_i) \subset A$. By construction

$$\left\|\Big(\sum|\tilde{\tau}_\alpha|^2\Big)^{1/2}\right\|_\infty = \left\|\Big(\sum|\tilde{\tau}_\alpha|^2\Big)^{1/2}\right\|_{L^\infty(\mathbf{T})} = \left\|\sum|\tau_\alpha|\right\|_\infty^{1/2} \leqq C$$

and similarly for the system $(\tilde{\tilde{\tau}}_\alpha)$.

Using (21), it therefore follows that

$$(20) \leqq CM^2(\log n)\|(\varphi_{i,j})\|_{M(A)}\Big(\sum_i M^2\|x_i\|^2\Big)^{1/2}\Big(\sum_j M^2\|y_j\|^2\Big)^{1/2}$$

which is (3) with $K = CM^4 \log n$. This completes the proof.

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